Face numbers of Engström representations of matroids

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April 16, 2013

Abstract

A classic problem in the theory of matroids is to find subspace arrangements, specifically hyperplane and pseudosphere arrangements, whose intersection posets are isomorphic to a prescribed geometric lattice. Engström recently showed how to construct an infinite family of such subspace arrangements, indexed by the set of finite regular CW complexes. In this note, we compute the face vectors of these representations (in terms of the face vectors of the indexing complexes) and give simple upper bounds on the total number of faces in these objects. In particular, we show, for a fixed rank, that the total number of faces in the Engström representation corresponding to a codimension one homotopy sphere arrangement is bounded above by a polynomial in the number of parallism classes of the matroid with degree one less than the rank.

1 Introduction

The *Topological Representation Theorem* for matroids, proved first by Folkman and Lawrence [5] for oriended matroids and then by Swartz [12] for all matroids, asserts that every (oriented) matroid can be realized by a codimension one pseudo/homotopy sphere arrangement. For a given matroid M, there are several constructions that provide such an arrangement representing M, e.g., see [1] and [7]. Each of these approaches takes as input the geometric lattice underlying M: the former utilizes flags along with tools presented in [4] whereas the latter utilizes homotopy colimits of diagrams of spaces [13].

While there are advantages to each approach, one significant difference is that the construction in [1] can have arbitrarily high dimension, for a fixed rank, while the construction in [7] keeps the dimension in direct correspondence with the rank function of the underlying matroid. The idea behind each is to glue homotopy spheres together into an arrangement either by including higher dimensional cells or mapping cylinders of an appropriate dimension. It is then natural to ask, from a computational perspective, how expensive it is to keep the dimension down? In other words, how many more faces are introduced with the mapping cylinders in our effort to control the dimension? In this note, we show that the answer to this question is not many when the rank is fixed. In particular, we prove that the total number of faces in an Engström representation of matroid M is a polynomial in the number of parallelism classes of M with degree at most one less than the rank of M. We also describe the asymptotic behavior of these numbers as rank increases.

2   Preliminaries

We begin with some basic definitions and theorems regarding geometric lattices and (oriented) matroids, see [5, 9, 14]; homotopy colimits of diagrams of spaces, see [13]; and Engström representations of matroids, see [7, 10].

2.1   Geometric Lattices and (Oriented) Matroids

Let E be a finite set. A *sign vector* is a vector $X \in \{+, 0, -\}^E$. The *zero set* of a sign vector is $z(X) = \{e \in E : X_e = 0\}$, and its support is $\underline{X} := \{e \in E : X_e \neq 0\}$. The *opposite* of a sign vector X is the vector $-X$ whose entries are the opposites of those in X; i.e.,

$$(-X)_e = \begin{cases} +, & \text{if } X_e = -, \\ -, & \text{if } X_e = +, \\ 0, & \text{if } X_e = 0. \end{cases}$$

The *composition* of two sign vectors X and Y is the sign vector $X \circ Y$ defined by

$$(X \circ Y)_e = \begin{cases} X_e, & \text{if } X_e \neq 0, \\ Y_e, & \text{otherwise.} \end{cases}$$

The *separation set* of X and Y is $S(X, Y) := \{e \in E : X_e = -Y_e \neq 0\}$.

An *oriented matroid* M consists of a finite set E, called the *ground set*, and a collection of *covectors* $\mathcal{L} \subseteq \{+, 0, -\}^E$ satisfying

- (L0) $0 \in \mathcal{L}$;
- (L1) if $X \in \mathcal{L}$, then $-X \in \mathcal{L}$;
- (L2) if $X, Y \in \mathcal{L}$, then $X \circ Y \in \mathcal{L}$; and
- (L3) if $X, Y \in \mathcal{L}$ and $e \in S(X, Y)$, then there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f = (Y \circ X)_f$ for all $f \notin S(X, Y)$.

If we ignore the sign data in $\mathcal{L}$, that is, if we consider only the set $L = \{z(X) : X \in \mathcal{L}\}$, then we obtain the underlying matroid of M. In this setting, (L1) becomes trivial and (L0), (L2), and (L3) become a set of axioms for matroids. In general, a *matroid* M consists of a finite set E, and a collection of *flats* $L \subseteq 2^E$, satisfying

- (F1) $E \in L$;
- (F2) if $X, Y \in L$, then $X \cap Y \in L$; and
- (F3) for every $X \in L$, the set of all $Y \backslash X$ where $X \subsetneq Y \in L$ and there is no $Z \in L$ such that $X \subsetneq Z \subsetneq Y$ forms a partition of $E \backslash X$.

For readers who prefer to think of matroids in terms of independent sets, the flats of M are the rank-maximal subsets of E, i.e. $\mathrm{rk}(X \cup e) > \mathrm{rk}(X)$ for any $e \in E \backslash X$. It is well known, see [3, 11], that L forms a graded *geometric lattice*, meaning

(1) L is semimodular (i.e. $\mathrm{rk}(p) + \mathrm{rk}(q) \geqslant \mathrm{rk}(p \wedge q) + \mathrm{rk}(p \vee q)$ for all $p, q \in L$) and

(2) every element of L is a join of atoms,

and, if M is oriented, the map $z : \mathcal{L} \to L$ is a cover-preserving, order-reversing surjection of $\mathcal{L}$ onto L, see [5, Proposition 4.1.13]. For every subset $X \subseteq E$, let

$$\overline{X} = \bigcap_{X \subseteq Y \in L} Y$$

denote the *closure* of X in M and define the *rank* of X to be the rank of $\overline{X}$ in L.

A *weak map* between matroids M and N is a function $\tau : E(M) \to E(N)$ such that $\mathrm{rk}_M(X) \geqslant \mathrm{rk}_N(\tau(X))$ for all $X \subseteq E(M)$. We leave it to the reader to check that every weak map $\tau : M \to N$ induces a (weakly rank-decreasing) order-preserving map $\overline{\tau} : L(M) \to L(N)$ given by $X \mapsto \overline{\tau(X)}$ for all $X \in L(M)$.

Lemma 2.1 ([10], Lemma 3). *If $\tau : M \to N$ is a surjective weak map, then $\overline{\tau} : L(M) \to L(N)$ is a surjective poset map.*

2.2 Topological Representations of (Oriented) Matroids

One of the most natural families of oriented matroids arises from real hyperplane arrangements. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be an essential arrangement of hyperplanes in $\mathbb{R}^r$, each of which contains the origin. Intersecting $\bigcup_i H_i$ with the unit sphere in a generic hyperplane $\mathcal{H} \subseteq \mathbb{R}^r$ yields a cell decomposition of S^{r-2}. Since each hyperplane H_i has a positive side and a negative side, we may associate a sign vector in $\{+, -, 0\}^n$ to each cell of this decomposition. The collection of such sign vectors, together with the zero vector, satisfy the covector axioms of an oriented matroid [5]. If an oriented matroid arises from a hyperplane arrangement in this way, we say that M is *realizable*.

The signed covectors of an oriented matroid form a lattice whose componentwise order relations are induced by declaring that $0 < +, -$. Folkman and Larence [8] proved that this lattice is isomorphic to the face poset of a cell decomposition of S^{r-2}. This cell decomposition is known as the *Folkman-Lawrence representation of M* [8], and we denote it as $\mathscr{S}(M)$. In the case that M is realizable, this decomposition is the natural one formed by intersecting the sphere with the corresponding hyperplane arrangement.

Without the orientation data, it is unknown how to find such a decomposition of S^{r-2} for prescribed matroid M. Instead, one can hope to construct a cell complex containing an arrangement of homotopy spheres whose intersection lattice matches $L(M)$. Engström [7] gave one such construction using diagrams of spaces, a convenient way to arrange topological spaces according to some prescribed combinatorial information, and homotopy colimits, the natural associated tool for gluing those spaces together with respect to the given information.

A *P-diagram of spaces*, $\mathcal{D}$, consists of the following data:

- a finite poset P,
- a CW complex $D(p)$ for every $p \in P$,
- a continuous map $d_{pq} : D(p) \to D(q)$ for every pair $p \geqslant q$ of P satisfying
  $d_{qr} \circ d_{pq}(x) = d_{pr}(x)$ for every triple $p \geqslant q \geqslant r$ of P and $x \in D(p)$.

To every diagram $\mathcal{D}$, we associate a topological space via a (homotopy) colimit. In our setting, the *homotopy colimit* of a diagram $\mathcal{D} : P \to \mathcal{Top}$ is the space

$$\mathrm{hocolim}_P \, \mathcal{D} = \coprod_{p \in P} (\Delta(P_{\leqslant p}) \times D(p)) \, / \sim$$

where $\sim$ is the transitive closure of the relation $(a, x) \sim (b, y)$ for each $a \in \Delta(P_{\leqslant p})$, $b \in \Delta(P_{\leqslant q})$, $x \in D(p)$ and $y \in D(q)$ if and only if $p \geqslant q$, $d_{pq}(x) = y$, and $a = b$.

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Example 2.2. Let P be the dual of the face poset (i.e, the face poset ordered by reverse inclusion) of the simplicial complex shown in Figure 1.
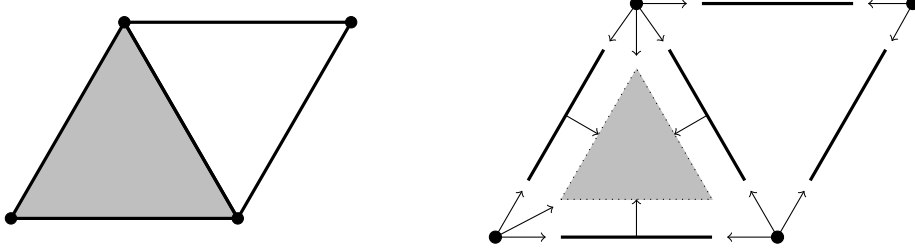

Figure 1: A simplicial complex (left) viewed as a diagram of spaces (right).

We form a diagram of spaces, $\mathcal{D}$ over P by declaring that $\mathcal{D}(p) = p$ for all faces $p \in \Delta$, and the inclusion maps d_{pq} are the natural inclusion maps. The diagram of spaces with the natural inclusion maps is shown in Figure 1 (left). The individual spaces $\Delta(P_{\leqslant p}) \times \mathcal{D}(p)$ are illustrated in Figure 2 (left) and the homotopy colimit of the diagram is illustrated in Figure 2 (right).
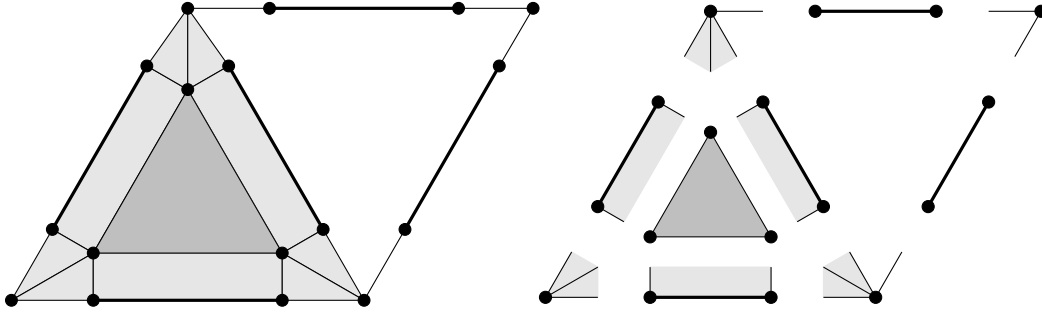

Figure 2: The homotopy colimit of Δ (left) decomposed as a union of open stars (right).

There is a natural notion of a structure-preserving map between diagrams of spaces. Let $\mathcal{D} : P \to \mathcal{T}op$ and $\mathcal{E} : Q \to \mathcal{T}op$ be diagrams of spaces over posets P and Q. A *morphism of diagrams* $(f, \alpha) : \mathcal{D} \to \mathcal{E}$ consists of a poset map $f : P \to Q$ together with a natural transformation α from $\mathcal{D}$ to $\mathcal{E} \circ f$. It is well known that morphisms of diagrams induce continuous maps between the corresponding homotopy colimits. In fact, these maps are completely explicit – if one writes each point in $\Delta(P_{\leqslant p}) \times D(p)$ as $(\lambda_1 p_1 + \cdots + \lambda_k p_k, x)$ where $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k = p$, $\lambda_i \geqslant 0$, $\sum_i \lambda_i = 1$, and $x \in D(p)$, then

$$f^\alpha(\lambda_1 p_1 + \cdots + \lambda_k p_k, x) = (\lambda_1 f(p_1) + \cdots + \lambda_k f(p_k), \alpha_p(x))$$

yields the desired map.

Remark 2.3. If $\mathcal{D}$ and $\mathcal{E}$ are diagrams of CW complexes with cellular maps and the maps in α are all cellular, then f^α is cellular as well.

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Let M be a rank r matroid and ℓ be a rank- and order-reversing poset map from $L(M)$ to B_r, the boolean lattice on $[r]$. For every locally finite, regular CW complex, X, and $\sigma \in B_r$ define a B_r-diagram $\mathcal{D}_X$ by

$$D_X(\sigma) = *_{i=1}^r \begin{cases} X & \text{if } i \in \sigma \\ \varnothing & \text{if } i \notin \sigma \end{cases}$$

with inclusion morphisms. This gives an $L(M)$-diagram $\mathcal{D}_X(M,\ell) := \mathcal{D}_X \circ \ell$. The *Engström representation* of a pair (M,ℓ), indexed by X, is the space

$$\mathscr{T}_X(M,\ell) := \operatorname{hocolim}_{L(M)} \mathcal{D}_X(M,\ell).$$

Engström [7, Theorem 3.7] showed that the homotopy type of $\mathscr{T}_X(M,\ell)$ is independent of ℓ and thus, it is often convenient to fix a standard choice of such map, e.g., $\hat{\ell} : L(M) \to B_r$ where $\hat{\ell}(p) = \{1, 2, \ldots, r - \operatorname{rk}(p)\}$, in which case we abbreviate $\mathscr{T}_X(M,\ell)$ to $\mathscr{T}_X(M)$.

Theorem 2.4 ([10], Corollary 3). *If $\tau : M \to N$ is a weak map, then $\overline{\tau}^\iota : \mathscr{T}_X(M) \to \mathscr{T}_X(N)$, where ι is the natural transformation of inclusions, is a continuous map.*

Corollary 2.5. *If $\tau : M \to N$ is a surjective weak map, then $\overline{\tau}^\iota : \mathscr{T}_X(M) \to \mathscr{T}_X(N)$ is a surjective cellular map.*

Proof. This follows easily from Lemma 2.1 and Remark 2.3. $\qquad\square$

3 Main Results

This is the main section of the paper where we compute the f-polynomial of the Engström representation, $\mathscr{T}_X(M)$, in terms of the f-polynomial of its indexing complex X. We also show that the total number of faces in $\mathscr{T}_X(M)$ is a polynomial in the number of parallelism classes of M with degree at most one less than the rank of M and describe the asymptotic behavior of these numbers as rank increases.

3.1 Face Polynomials of Engström Representations

The f-*polynomial* of a (finite) CW complex Δ is the polynomial $f_\Delta(t) := \sum_{i \geqslant 0} f_{i-1}(\Delta)t^i$, where $f_{i-1}(\Delta)$ enumerates the number of $(i-1)$-dimensional faces in Δ. In particular, notice that $f_{-1}(\Delta) = 1$, corresponding to the empty face, for any nonempty complex Δ. We make use of the following three standard formulas for computing f-polynomials: If Δ and Γ are CW complexes, then

$$f_{\Delta * \Gamma}(t) = f_\Delta(t) \cdot f_\Gamma(t), \tag{1}$$

$$f_{\Delta \times \Gamma}(t) = \frac{(f_\Delta(t) - 1) \cdot (f_\Gamma(t) - 1)}{t} + 1, \tag{2}$$

$$f_{\Delta \sqcup \Gamma}(t) = f_\Delta(t) + f_\Gamma(t) - 1. \tag{3}$$

Theorem 3.1. *For a given CW complex, X, and matroid, M,*

$$f_{\mathscr{T}_X(M)}(t) = 1 + \sum_{p \in L(M)} \frac{(f_{\Delta_M^\circ(p)}(t) - 1) \cdot (f_X^{\operatorname{corank}(p)}(t) - 1)}{t}$$

where $\Delta_M^\circ(p)$ is the open star of p in $\Delta(L(M)_{\leqslant p})$.

Proof. The Engström representation $\mathscr{T}_X(M)$ arises naturally as a quotient of the space Y that is a disjoint union of spaces indexed by the elements $p \in L(M)$. Each such p contributes a component to Y that is the product of $\Delta(L(M)_{\leqslant p})$ with a $(\text{corank}(p))$-fold join of the space X. The f-polynomial of this space is easily computed using formulas (1)-(3) as

$$f_Y(t) = \sum_{p \in L(M)} \left(\frac{(f_{\Delta(L(M)_{\leqslant p})}(t) - 1) \cdot (f_X^{\text{corank}(p)}(t) - 1)}{t} + 1 \right) - (|L(M)| - 1).$$

Thus, the only difficulty in computing the f-polynomial of $\mathscr{T}_X(M)$ is accounting for the quotient $\sim$. One could proceed, naïvely, by sieving out the over-counted cells identified by $\sim$, but it is simpler to observe that $\mathscr{T}_X(M)$ can be decomposed nicely into a disjoint union of half-open spaces. Since every vertex $q \in \Delta(L(M)_{\leqslant p}) \backslash p$ will be identified to itself in $\Delta(L(M)_{\leqslant q})$, it suffices to consider only the space $\Delta_M^\circ(p)$ that is the union of all open cells in $\Delta(L(M)_{\leqslant p})$ whose closures contain p. The desired result follows by replacing $\Delta(L(M)_{\leqslant p})$ by $\Delta_M^\circ(p)$ for each $p \in L(M)$. $\square$

Remark 3.2. For $\hat{1} = E(M) \in L(M)$, $D_X(\hat{1}) = \varnothing$ which makes the summand in Theorem 3.1 equal to zero. Hence, it suffices to take the sum over $L(M) \backslash \hat{1}$.

Example 3.3. Let $U_{r,n}$ denote the rank r *uniform matroid* on n elements , i.e., the matroid on $[n]$ whose flats consist of $[n]$ and all subsets of size at most r, and let $L = L(U_{r,n})$ for some fixed $r \leqslant n \in \mathbb{N}$. We note that the subposet of L of rank less than r is isomorphic to that of B_n and for any $p, q \in L$ with $\text{rk}(p) = \text{rk}(q) = i$,

$$\Delta_L^\circ(p) \times D_X(p) = \Delta_L^\circ(q) \times D_X(q) = \Delta_{B_i}^\circ(\hat{1}) \times X^{*(r-i)}.$$

To compute $f_k(\Delta_{B_i}^\circ(\hat{1}))$, we count the number of length k chains in B_i that contain $\hat{1}$. There are $(k+1)! \cdot S(i, k+1) + k! \cdot S(i, k) = k! \cdot S(i+1, k+1)$ such chains, where $S(i, j)$ is the Stirling number of the second kind, because every partition of $[i]$ with j parts yields $j!$ chains of length $j - 1$ not containing $\hat{0} = \varnothing$ in B_i and $j!$ chains of length j that do contain $\hat{0} = \varnothing$.

Bringing this all together for a given X, Theorem 3.1 asserts that

$$f_{\mathscr{T}_X(U_{r,n})}(t) = 1 + \sum_{i=0}^{r} \binom{n}{i} \cdot \frac{(S_i(t) - 1) \cdot (f_X^{r-i}(t) - 1)}{t},$$

where $S_i(t) = 1 + \sum_{k=0}^{i} k! S(i+1, k+1) t^{k+1}$.

Remark 3.4. Since every rank r matroid on n elements is a surjective weak map image of $U_{r,n}$, Corollary 2.5 implies that the formula in Example 3.3 gives upper bounds for the f-polynomials of the Engström representations of any matroid.

3.2 The Total Number of Faces of an Engström Representation

For the remainder of this section, we restrict our attention to the Engström representations where $X = S^0$, i.e., the codimension one homotopy sphere arrangements and most natural objects to compare with the Folkman-Lawrence representations of oriented matroids.

Corollary 3.5. *If M is a rank r matroid on n elements, then*

$$f_{\mathscr{T}_{S^0}(M)}(1) \leqslant f_{\mathscr{T}_{S^0}(U_{r,n})}(1) = 1 + \sum_{i=0}^{r} \left[\binom{n}{i} \cdot 2F_i \cdot (3^{r-i} - 1) \right],$$

where $F_i = \sum_{k=0}^{i} k! S(i,k)$ is the ith Fubini number.

Proof. The inequality follows immediately from Remark 3.4. The equality on the right follows from Example 3.3 by setting $f_{S^0}(t) = 1 + 2t$, evaluating at $t = 1$, and observing that

$$\sum_{k=0}^{i} k! \cdot S(i+1, k+1) = \sum_{k=0}^{i} ((k+1)! \cdot S(i, k+1) + k! \cdot S(i,k)) = 2 \cdot \sum_{k=0}^{i} k! \cdot S(i,k).$$

$\square$

Dong [6, Corollary 2.7] showed an oriented matroid is uniform if and only if $\mathrm{rk}(X) = r - |z(X)|$ for every nonzero covector $X \in \mathcal{L}$. As a consequence, the support of a covector of rank $k + 1$ in $\mathcal{L}(U_{r,n})$ has cardinality $n - r + k + 1$, which gives the following upper bound on the number of faces in the Folkman-Lawrence representation of the uniform matroid.

Lemma 3.6. *For every $r \leqslant n \in \mathbb{N}$,*

$$f_{\mathscr{S}(U_{r,n})}(1) \leqslant 1 + \sum_{k=0}^{r-1} \left[\binom{n}{r-k-1} \cdot 2^{n-r+k+1} \right] = 1 + \sum_{i=0}^{r-1} \left[\binom{n}{i} \cdot 2^{n-i+2} \right].$$

When r is fixed, Corollary 3.5 asserts that $f_{\mathscr{T}_{S^0}(M)}$ is bounded above by a polynomial in n of degree at most $r - 1$, but when r is allowed to vary, these numbers grow rather quickly. Indeed, this is to be expected since, by Lemma 3.6, the total number of faces grows exponentially with r even in the optimal case of the Folkman-Lawrence representations. However, these numbers grow substantially faster for the Engström representations since

$$\frac{2F_k \cdot (3^{r-k} - 1)}{2^{n-k+2}} \approx C_{r,n} \cdot k! \left(\frac{2}{3 \log 2} \right)^k,$$

as k gets large [2] where $C_{r,n} = \dfrac{3^r}{2^{n+2} \log 2}$. This prompts us to ask the following:

Question 3.7. *Are there topological representations of matroids whose total face numbers grow more closely to those of the Folkman-Lawrence representations as the rank gets large?*

Remark 3.8. One can replace the function $3^{r-i} - 1$ in Corollary 3.5 with $2(r - i)$ by replacing $D_{S^0}(p)$ with the standard cell structure on S^{r-i-1} consisting of two cells in each dimension, but compared to the growth of Fubini numbers, it is not clear that such an alteration yields a meaningful improvement for large r.

4   Appendix

We conclude with an application of Theorem 3.1 to the Fano matroid.

Example 4.1. Recall that the Fano plane, F, is a rank three matroid on seven elements, as depicted in Figure 3, and that every $L(F)_{\leqslant p} \cong L(F)_{\leqslant q}$ for every $p, q \in L(F)$ with $\mathrm{rk}(p) = \mathrm{rk}(q)$.
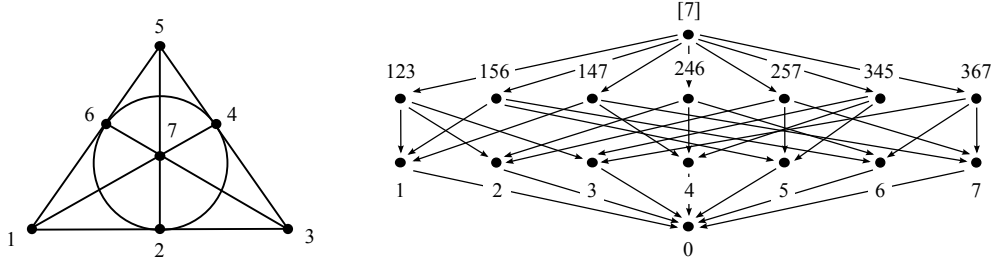

Figure 3: The Fano plane (left) and its lattice of flats (right).

We leave it to the reader to verify that

$$f_{\Delta_F^{\circ}(p)}(t) = \begin{cases} 1 + t & \mathrm{rk}(p) = 0, \\ 1 + t + t^2 & \mathrm{rk}(p) = 1, \\ 1 + t + 4t^2 + 3t^3 & \mathrm{rk}(p) = 2. \end{cases}$$

Plugging this into Theorem 3.1, along with $f_{S^0}(t) = 1 + 2t$, we get that

$$\begin{aligned} f_{\mathscr{T}_{S^0}(F)}(t) &= 1 + 1(6t + 12t^2 + 8t^3) + 7(4t + 8t^2 + 4t^3) + 7(2t + 8t^2 + 6t^3) + 1(0) \\ &= 1 + 48t + 124t^2 + 78t^3. \end{aligned}$$

A portion of $\mathscr{T}_{S^0}(F)$, namely $\mathrm{hocolim}_{L(F)_{>\varnothing}} \mathcal{D}_{S^0}$, is illustrated in Figure 4.
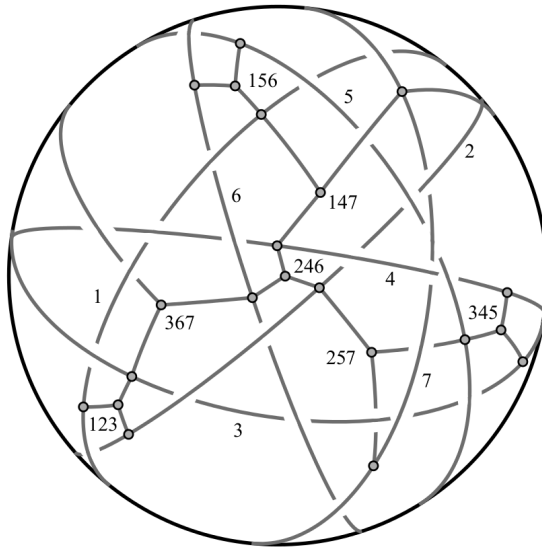

Figure 4: A portion of the Engström representation of F with $X = S^0$.

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